Recent Trends in the Design of Biorthogonal Modulated Filter Banks

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Abstract

Biorthogonal modulated filter banks, when compared to paraunitary ones, provide the advantage that the overall system delay can be chosen independently of the filter length, thus resulting in low delay filter banks. They have recently been studied by several authors. In the paper, we connect different design methods (quadratic constrained least-squares optimization, cascade of sparse self-inverse matrices) and describe advantages of the a factorization into Zero-Delay and Maximum-Delay matrices (structure-inherent perfect reconstruction, no DC leakage of the filter bank, low implementation cost)

1 Introduction

Modulated filter banks have been studied extensively in literature within the last 10 years. They have shown to provide a very efficient implementation based on a prototype filter and a fast transform. The most popular modulation scheme is cosine modulation. However, other modulation schemes based on Discrete Fourier Transform (DFT) also exist. Historically, the first modulated filter banks with perfect reconstruction were designed such as to be paraunitary [PB86, RT91, KV92, Mau93, NK96]. In this special case, the impulse responses of the synthesis filters are flipped versions of the analysis ones and all filters are derived from one common prototype. However, since the overall system delay of the filter bank is directly related to the filter length, the desired features (a) a high stopband attenuation of the filters and (b) a short overall system delay are contradictory. This problem has partly been overcome with the design of low-delay biorthogonal filter banks, where the overall system delay can be chosen independently (within some fundamental limits) of the filter length and the number of subbands. In this class of filter banks, the synthesis filters are no longer flipped versions of the analysis filters and the analysis and synthesis filters may be derived from different prototypes.

Two principal approaches can be observed for the design of biorthogonal cosine-modulated filter banks with perfect reconstruction. The approach by Schuller et al. [SS96, Sch96, SK97] uses filter bank realizations that structurally guarantee perfect reconstruction for arbitrary system delays. It is mainly based on a factorization of the analysis polyphase matrix into a transform and special sparse matrices which are easy to invert. The inverse matrices are then used on the synthesis side. The prototype filters are derived by nonlinear optimization methods. Completeness of the factorization has been shown for all contiguous prototype filters, i.e. prototype filters that do not have any zero taps within their region of support.
On the other hand, Nguyen et al. explicitly derive perfect reconstruction (PR) constraints for the polyphase components of the prototype filters [Ngu92, NH96, HKN96]. The filter design is carried out by a quadratic-constrained least-squares optimization (QCLS algorithm) using the stopband energy of the prototypes' frequency responses as a cost function and the PR conditions as constraints. In [HKN96] it has also been shown that the same PR constraints also hold true for a DCT-II modulation scheme as proposed in [LV95].
Both approaches provide certain advantages. Since in the latter case, the PR constraints are directly formulated, it can be easily verified whether or not given prototype filters yield perfect reconstruction. Necessary relations between the analysis and synthesis prototype filters are also stated [HKN96]. Furthermore, it can be shown that for certain combinations of filter lengths and overall system delay, some polyphase filters can only have one non-zero coefficient. This case is not treated in the factorization proposed in [Sch96, SK97]. However, the factorization approach from Schuller et al. offers many advantages concerning the implementation of the filter bank. First of all, the structure automatically guarantees PR. This also holds true when using integer-valued coefficients, because the same factorization coefficients are used on the analysis and synthesis side. But even when using coefficients with infinite precision, it turns out that the implementation cost is nearly halved when compared to the direct realization of the polyphase filters as assumed by Nguyen et al.. Another advantage of the approach is that it has been extended to the case of time-varying filter banks [Sch97] without much effort. Which of the two optimization methods (unconstrained and non-linear or constrained and quadratic) results in better filter designs, highly depends on the chosen optimization procedures and the complexity of the problem (i.e. filter length, number of subbands, etc.).
In this paper, we connect both approaches by showing that for the modulation scheme considered by Nguyen et al., a factorization being similar the one proposed by Schuller et al. exists. The factorization is derived directly from the PR constraints on the polyphase components of the prototype filters and also treats the case that some polyphase filters contain coefficients being equal to zero. Instead of dealing with size $M \times M$ matrices as in [Sch96, SK97] we just have to deal with size 2×2 matrices and realize $\lfloor M/2 \rfloor$ of them in parallel. Using this factorization, the implementation cost can be significantly reduced. Furthermore, we show how to include certain useful filter bank features in the implementation. Such features can be

- same prototype for analysis and synthesis
- prototype filters with specified zeros at certain frequencies in order to yield filter banks without DC leakage
- integer coefficient prototype filters

- time-varying filter banks

The outline of the paper is as follows. In Section 2, we recall the PR constraints of the cosine-modulated filter banks as derived by Nguyen et al. In Section 3, we derive how to realize this filter bank using Zero-Delay and Maximum-Delay matrices starting from the PR constraints. Section 4 shows that we can easily design filter banks with identical analysis and synthesis prototype filter by imposing some constraints on the first Delay matrix. Filter banks without DC leakage are important when treating e.g. images. This feature can also be obtained when choosing the first matrix of the factorization appropriately, as will be derived in Section 5. Section 6 compares the implementation cost of the new factorization with a direct implementation of the polyphase filters and in Section 7 we explain how to use the structure-inherent PR property of the factorization in order to design VLSI efficient prototype filters. In Section 8, we shortly sketch the extension of the framework to time-varying filter banks and summarize the results of the paper in Section 9.

2 Cosine-Modulated Filter Banks with Perfect Reconstruction

In this section, we recall the main steps of the derivation of PR constraints from [HKN96, NH96] for biorthogonal cosine-modulated filter banks. The filter bank structure is shown in Figure 1. The analysis filter bank consists of M analysis filters of length N_h with impulse responses $h_k(n)$, $k = 0, \ldots, M-1$, $n = 0, \ldots, N_h - 1$, and subsequent downsampling by M. The input signal is denoted as $x(n)$. The subband signals are $y_k(m)$, $k = 0, \ldots, M-1$, where m is the time index at the reduced sampling rate. The synthesis filter bank consists of upsamplers by M, followed by M synthesis filters with impulse responses $f_k(n)$, $k = 0, \ldots, M-1$, $n = 0, \ldots, N_f - 1$. The outputs of these filters are summed to form the reconstructed signal $\hat{x}(n)$.

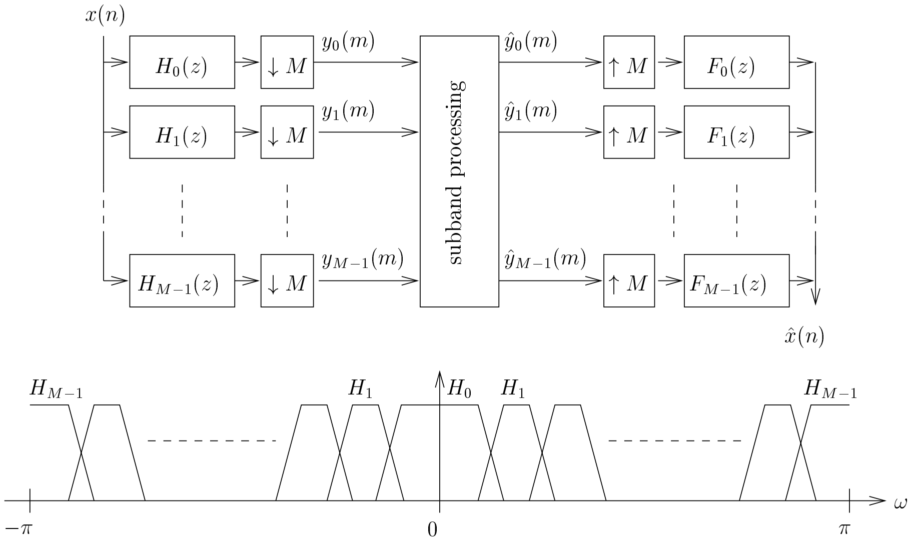

Figure 1: Biorthogonal cosine-modulated filter bank

The generation of the M analysis and synthesis filters , $H_k(z)$ and $F_k(z)$, respectively, from the lowpass prototype filters $H(z)$ and $F(z)$ has been chosen according to

$$h_k(n) = 2h(n)\cos\left(\frac{\pi}{M}(k+0.5)(n-D/2)+\theta_k\right), \quad n=0,1,\dots,N_h-1 \tag{1}$$

$$f_k(n) = 2f(n)\cos\left(\frac{\pi}{M}(k+0.5)(n-D/2)-\theta_k\right), \quad n=0,1,\dots,N_f-1 \tag{2}$$

with $\theta_k = (-1)^k\frac{\pi}{4}$, $k=0,\dots,M-1$ and $D = 2sM+d$, $0 \le d < 2M$. Herein, D describes the overall system delay of the filter bank, assuming that the subband signals are directly passed from the analysis to the synthesis bank.
The filter bank provides perfect reconstruction (PR) if the output signal is a delayed version of the input signal, $\hat{x}(n) = x(n-D)$. The derivation of the PR constraints is based on the polyphase representation of the filter bank as shown in Figure 2.

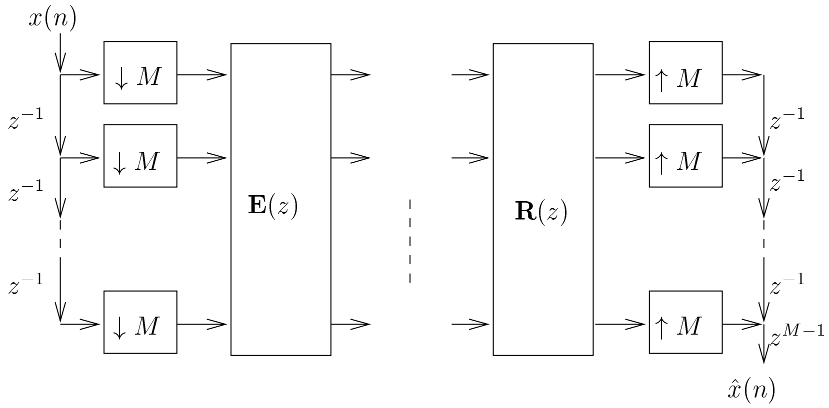

Figure 2: Filter bank realization using polyphase matrices

In [HKN96] it has been shown that the analysis and synthesis filter polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$, respectively, write

$$\mathbf{E}(z) = \mathbf{C}_1\begin{bmatrix}\mathbf{g}_0(-z^2)\\ z^{-1}\mathbf{g}_1(-z^2)\end{bmatrix}, \qquad \mathbf{R}(z) = \begin{bmatrix}z^{-1}\mathbf{k}_1(-z^2) & \mathbf{k}_0(-z^2)\end{bmatrix}\mathbf{C}_2^t \tag{3}$$

with

$$[\mathbf{C}_1]_{k,\ell} = 2\cos\left((k+0.5)\frac{\pi}{M}(\ell-\frac{D}{2})+\theta_k\right), \; 0\le k<M,\; 0\le \ell<2M \tag{4}$$

$$[\mathbf{C}_2]_{k,\ell} = 2\cos\left((k+0.5)\frac{\pi}{M}(2M-1-\ell-\frac{D}{2})-\theta_k\right), \; 0\le k<M,\; 0\le \ell<2M \tag{5}$$

$$\begin{aligned}
\mathbf{g}_0(-z^2) &= \mathrm{diag}[G_0(-z^2),\dots,G_{M-1}(-z^2)]\\
\mathbf{g}_1(-z^2) &= \mathrm{diag}[G_M(-z^2),\dots,G_{2M-1}(-z^2)]\\
\mathbf{k}_0(-z^2) &= \mathrm{diag}[K_{M-1}(-z^2),\dots,K_0(-z^2)]\\
\mathbf{k}_1(-z^2) &= \mathrm{diag}[K_{2M-1}(-z^2),\dots,K_M(-z^2)]
\end{aligned}$$

where $G_\ell(z)$ and $K_\ell(z)$, $\ell = 0, \ldots, 2M-1$ denote the ℓ-th type-1 polyphase component of the analysis and synthesis prototype filter, respectively:

$$H(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} G_\ell(z^{2M}), \qquad F(z) = \sum_{\ell=0}^{2M-1} z^{-\ell} K_\ell(z^{2M}) \tag{6}$$

Using the upper equations, the overall polyphase matrix $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$ writes

$$\mathbf{P}(z) = \begin{bmatrix} z^{-1}\mathbf{k}_1(-z^2) & \mathbf{k}_0(-z^2) \end{bmatrix} \mathbf{C}_2^t \mathbf{C}_1 \begin{bmatrix} \mathbf{g}_0(-z^2) \\ z^{-1}\mathbf{g}_1(-z^2) \end{bmatrix} \tag{7}$$

For perfect reconstruction and an overall system delay of D samples, $\mathbf{P}(z)$ has to satisfy [Vai93]:

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z) = \begin{cases} z^{-(2s-1)} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-1-d} \\ z^{-1}\mathbf{I}_{d+1} & \mathbf{0} \end{bmatrix}, & 0 \le d < M \\ z^{-2s} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{2M-1-d} \\ z^{-1}\mathbf{I}_{d+1-M} & \mathbf{0} \end{bmatrix}, & M \le d < 2M \end{cases} \tag{8}$$

From (7) and (8), the following constraints can be derived on the polyphase filters, see [HKN96]:

PR constraints for $0 \le d < M$:

The PR constraints for $0 \le d < M$, as derived in [HKN96], can be summarized as follows (recall that the overall system delay writes $D = 2sM + d$):

$$K_\ell(z) = \lambda_\ell z^{-a_\ell} G_\ell(z), \quad K_{\ell+M}(z) = \lambda_\ell z^{-a_\ell} G_{\ell+M}(z), \quad 0 \le \ell < M, \quad \ell \ne \frac{M+d}{2} \tag{9}$$

with $a_\ell \in I\!N$ being an additional zero-padding and λ_ℓ a scaling, and:

$$(i) \quad G_\ell(z)K_{d-\ell}(z) + z^{-1}G_{M+\ell}(z)K_{M+d-\ell}(z) = \frac{z^{-s}}{2M}, \qquad 0 \le \ell \le d \tag{10}$$

$$(ii) \quad G_\ell(z)K_{2M+d-\ell}(z) + G_{M+\ell}(z)K_{M+d-\ell}(z) = \frac{z^{-(s-1)}}{2M}, \qquad d < \ell < 2M, \ \ell \ne \frac{M+d}{2} \tag{11}$$

$(iii) \quad \ell = (M+d)/2:$

$$G_{\frac{M+d}{2}}(z)K_{\frac{3M+d}{2}}(z) = \frac{z^{-(s-1)}}{4M} \qquad K_{\frac{M+d}{2}}(z),\ G_{\frac{3M+d}{2}}(z) \ \text{arbitrary} \qquad \text{for } s \text{ odd} \tag{12}$$

$$G_{\frac{3M+d}{2}}(z)K_{\frac{M+d}{2}}(z) = \frac{-z^{-(s-1)}}{4M} \qquad K_{\frac{3M+d}{2}}(z),\ G_{\frac{M+d}{2}}(z) \ \text{arbitrary} \qquad \text{for } s \text{ even} \tag{13}$$

PR constraints for $M \leq d < 2M$:
The PR constraints for $M \leq d < 2M$ can be expressed in a similar way:

$$K_\ell(z) = \lambda_\ell z^{-a_\ell} G_\ell(z), \quad K_{\ell+M}(z) = \lambda_\ell z^{-a_\ell} G_{\ell+M}(z), \quad 0 \leq \ell < M, \quad \ell \neq \frac{d-M}{2} \tag{14}$$

and

$$(i) \quad G_\ell(z)K_{d-\ell}(z) + G_{M+\ell}(z)K_{d-\ell-M}(z) = \frac{z^{-s}}{2M} \qquad 0 \leq \ell \leq d-M,\ \ell \neq \frac{d-M}{2} \tag{15}$$

$$(ii) \quad G_\ell(z)K_{d-\ell}(z) + z^{-1}G_{\ell+M}(z)K_{M+d-\ell}(z) = \frac{z^{-s}}{2M} \qquad d-M < \ell < 2M \tag{16}$$

$(iii) \quad \ell = (d-M)/2:$

$$G_{\frac{M+d}{2}}(z)K_{\frac{d-M}{2}}(z) = \frac{-z^{-s}}{4M} \qquad K_{\frac{M+d}{2}}(z),\ G_{\frac{d-M}{2}}(z) \text{ arbitrary} \qquad \text{for } s \text{ odd} \tag{17}$$

$$G_{\frac{d-M}{2}}(z)K_{\frac{M+d}{2}}(z) = \frac{z^{-s}}{4M} \qquad K_{\frac{d-M}{2}}(z),\ G_{\frac{M+d}{2}}(z) \text{ arbitrary} \qquad \text{for } s \text{ even} \tag{18}$$

Some remarks on the PR constraints: From (9) and (14), it can be seen that the polyphase components of the analysis and synthesis prototype filters are strictly connected. I.e., they have to be equal up to the scale factors λ_ℓ and the delays a_ℓ. The value d of the overall system delay determines which polyphase filters are connected in the PR constraints (10)-(13) and (15)-(18), while s determines the delay on the right-hand side of the upper equations. Note that for $d = 2M-1$ and M being even, all PR constraints are given by (14) and (15).

3 Filter Bank Realization using Zero-Delay and Maximum-Delay Matrices

A straightforward implementation of the biorthogonal cosine-modulated filter bank can be derived from the polyphase matrices in (3). As shown in Figure 3, the input signal is split into M polyphase components. These components are fed into the $2M$ polyphase filters $G_\ell(-z^2)$. Their outputs are then transformed by the $2M \times M$ transform matrix $\mathbf{C}_1$ and yield the vector of M subband signals. On the synthesis side, mainly the inverse steps are performed.

From the filter bank realization in Figure 3, the following ideas arise for a more efficient realization: First, the polyphase matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ are of size $M \times 2M$ and not of size $M \times M$ as in [Mal92, Sch96], which means that the complexity might be reduced by exploiting the properties of the modulation matrices. Second, the polyphase components $G_\ell(-z^2)$ and $G_{\ell+M}(-z^2)$ are fed with the same input signal, so that one could think about realizing both analysis polyphase filters jointly. Accordingly, on the synthesis side, the outputs of $K_\ell(-z^2)$ and $K_{\ell+M}(-z^2)$ are added, and both synthesis polyphase filters may also be realized jointly. In the following, we show how these ideas result in an efficient realization. For simplicity, we only consider the case where the delay parameter d is in the range $M \leq d < 2M$. Due to the different PR constraints (15)-(18) we

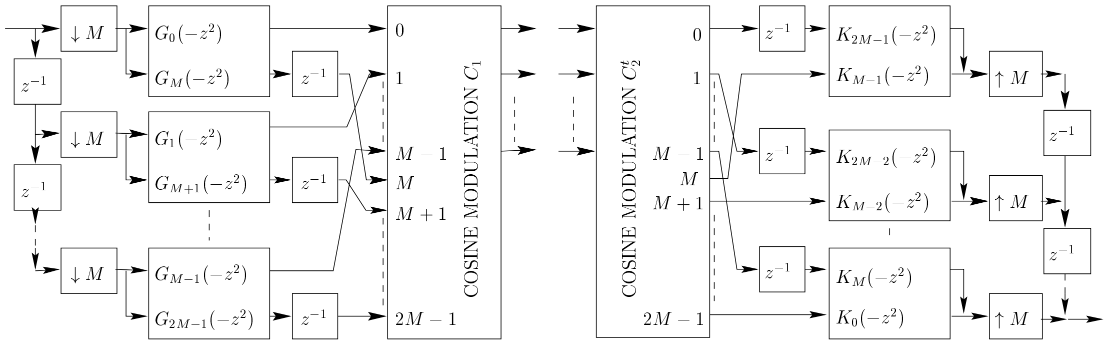

Figure 3: Filter bank realizations using polyphase filters $G_\ell(z)$ and $K_\ell(z)$, respectively, and cosine transform

have to treat the cases (i) to (iii) separately. The derivations for $0 \le d < M$ can be performed in an analog way.

Case (i), $0 \le \ell \le d - M$, $\ell \ne \frac{d-M}{2}$. In the following, we regard the range $0 \le \ell < (d-M)/2$. The results for $(d-M)/2 < \ell \le d-M$ are the same as the former ones when substituting ℓ by $d - M - \ell$.
By having a closer look at the modulation matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ from (4)-(5) one can verify that

$$[\mathbf{C}_1]_{k,d-M-\ell} = (-1)^s [\mathbf{C}_1]_{k,\ell} \qquad [\mathbf{C}_2]_{k,M-1-\ell} = (-1)^s [\mathbf{C}_2]_{k,2M-1-d+\ell} \tag{19}$$

$$[\mathbf{C}_1]_{k,\ell+M} = (-1)^{s-1} [\mathbf{C}_1]_{k,d-\ell} \qquad [\mathbf{C}_2]_{k,3M-1-d+\ell} = (-1)^{s-1} [\mathbf{C}_2]_{k,2M-1-\ell} \tag{20}$$

Starting from the analysis and synthesis polyphase matrices in (3) and considering subsystems $\mathbf{E}_\ell^{(i)}(z)$ and $\mathbf{R}_\ell^{(i)}(z)$ of the polyphase matrices that contain the columns of the modulation matrices which are connected by the upper equations, we obtain for the subsystem of the analysis polyphase matrix:

$$\mathbf{E}_\ell^{(i)}(z) = \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell-M} & c^1_{0,\ell+M} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell-M} & c^1_{1,\ell+M} & c^1_{1,d-\ell} \\ \vdots & & & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell-M} & c^1_{M-1,\ell+M} & c^1_{M-1,d-\ell} \end{bmatrix} \begin{bmatrix} G_\ell(-z^2) & 0 \\ 0 & G_{d-\ell-M}(-z^2) \\ z^{-1}G_{\ell+M}(-z^2) & 0 \\ 0 & z^{-1}G_{d-\ell}(-z^2) \end{bmatrix}$$

$$= \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell} \\ \vdots & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell} \end{bmatrix} \underbrace{\begin{bmatrix} G_\ell(-z^2) & (-1)^s G_{d-\ell-M}(-z^2) \\ (-1)^{s-1} z^{-1} G_{\ell+M}(-z^2) & z^{-1} G_{d-\ell}(-z^2) \end{bmatrix}}_{:=\mathbf{G}_\ell^{(i)}} \tag{21}$$

with $c^1_{k,\ell} = [\mathbf{C}_1]_{k,\ell}$. Similarly, the subsystem $\mathbf{R}_\ell^{(i)}(z)$ of the synthesis polyphase matrix writes

$$\mathbf{R}_\ell^{(i)}(z) =$$

$$\begin{bmatrix} z^{-1}K_{d-\ell}(-z^2) & 0 & K_{d-\ell-M}(-z^2) & 0 \\ 0 & z^{-1}K_{\ell+M}(-z^2) & 0 & K_\ell(-z^2) \end{bmatrix} \begin{bmatrix} c^2_{0,2M-1-d+\ell} & \cdots & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,M-1-\ell} & & c^2_{M-1,M-1-\ell} \\ c^2_{0,3M-1-d+\ell} & & c^2_{M-1,3M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \cdots & c^2_{M-1,2M-1-\ell} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} z^{-1}K_{d-\ell}(-z^2) & (-1)^{s-1}K_{d-\ell-M}(-z^2) \\ (-1)^s z^{-1}K_{\ell+M}(-z^2) & K_\ell(-z^2) \end{bmatrix}}_{:=\mathbf{K}_\ell^{(i)}} \begin{bmatrix} c^2_{0,2M-1-d+\ell} & & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \cdots & c^2_{M-1,2M-1-\ell} \end{bmatrix} \quad (22)$$

with $c^2_{k,\ell} = [\mathbf{C}_2]_{k,\ell}$. The analysis and synthesis filter bank now can be realized as shown in Figure 4.

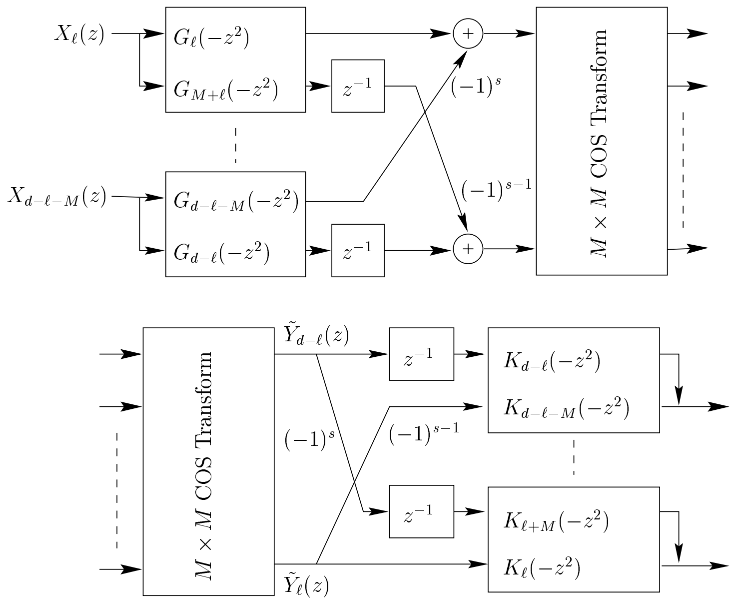

Figure 4: New realization of analysis and synthesis filter bank

By these steps, we have achieved two points: first of all, we have suppressed half the columns of the analysis and synthesis modulation submatrices, resulting in a lower modulation cost. Note, however, that some further arrangement of the rows and columns of the new modulation matrices is needed before obtaining a form that can be realized by fast DCT. The second point deals with a more efficient realization of the polyphase filtering part. From Figure 4 it can be seen that we do not have to calculate the output of all four polyphase filters in the figure but only the sum of two outputs of the analysis or synthesis polyphase filters.

When calculating $\mathbf{K}_\ell^{(i)}\mathbf{G}_\ell^{(i)}$ with $\mathbf{K}_\ell^{(i)}$ and $\mathbf{G}_\ell^{(i)}$ from (22) and (21), respectively, substituting $-z^2$ by z in the result and comparing the four entries of the matrix with the constraints for perfect reconstruction (14) and (15), it can be verified that the following relationship has to hold true for PR:

$$\mathbf{K}_\ell^{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{G}_\ell^{(i)} = \frac{z^{-1}(-z^{-2})^s}{2M} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 \leq \ell \leq d-M, \quad \ell \neq \frac{d-M}{2} \tag{23}$$

This relation can now be used in order to design new prototype filters. The filter design consists of the following steps:

- Starting point: We start with length-1 entries for the polyphase components in $\mathbf{G}_{\ell,0}^{(i)}$ and $\mathbf{K}_{\ell,0}^{(i)}$ that satisfy

$$\mathbf{K}_{\ell,0}^{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{G}_{\ell,0}^{(i)} = \frac{z^{-1}}{2M} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

  The following solution exists for the starting point:

$$\mathbf{G}_{\ell,0}^{(i)} = \begin{bmatrix} g_0 & g_1 \\ z^{-1}g_2 & z^{-1}g_3 \end{bmatrix}, \quad \mathbf{K}_{\ell,0}^{(i)} = \frac{1}{2M} \cdot \frac{1}{g_0 g_3 - g_1 g_2} \begin{bmatrix} g_3 z^{-1} & -g_1 \\ -g_2 z^{-1} & g_0 \end{bmatrix} \tag{25}$$

- Increasing the filter length: In order to increase the prototype filter length, we have to increase the lengths of the polyphase filters. This can be done by replacing the identity matrix in (24) by so called Zero-Delay matrices that increase the filter length, but not the overall system delay. Possible solutions for the Zero-Delay matrices are:

$$\mathbf{A}_\ell^{-1}\mathbf{A}_\ell = \begin{bmatrix} 1 & 0 \\ -a_\ell z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_\ell z^{-1} & 1 \end{bmatrix}, \quad \mathbf{B}_\ell^{-1}\mathbf{B}_\ell = \begin{bmatrix} 1 & -b_\ell z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_\ell z^{-1} \\ 0 & 1 \end{bmatrix} \tag{26}$$

  When being applied, the Low-Delay matrices maintain the structure that is inherent to $\mathbf{G}_\ell^{(i)}$ containing even powers of z in its first row and odd powers of z in its second row.

  Note that not all possible cascades of Low-Delay matrices increase the length of all polyphase filters equally. However, the polyphase filters in $\mathbf{G}_{\ell,k}^{(i)}$ all can have $k+1$ non-zero coefficients when using the following realization:

$$\mathbf{G}_{\ell,k}^{(i)} = \prod_{i=1}^{k} (\mathbf{A}_{\ell,i}\mathbf{B}_{\ell,i})\mathbf{G}_{\ell,0}^{(i)} \tag{27}$$

- Increasing the filter length and the overall system delay: So far, we have only increased the filter length, but the delay of the product $\mathbf{K}_{\ell,k}^{(i)}\mathbf{G}_{\ell,k}^{(i)}$ remains the same as in (24). However, we can write the matrix product as

$$\mathbf{K}_{\ell,k}^{(i)} \begin{bmatrix} -z^{-2} & 0 \\ 0 & -z^{-2} \end{bmatrix}^{s} \mathbf{G}_{\ell,k}^{(i)} = \frac{z^{-1}(-z^{2})^{-s}}{2M} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

and replace each of the s delay matrices by one of the following Maximum-Delay matrices:

$$-z^{-2}\mathbf{C}_{\ell}^{-1}\mathbf{C}_{\ell} = \begin{bmatrix} 0 & -z^{-1} \\ -z^{-1} & c_{\ell} \end{bmatrix} \begin{bmatrix} c_{\ell} & z^{-1} \\ z^{-1} & 0 \end{bmatrix}, \; -z^{-2}\mathbf{D}_{\ell}^{-1}\mathbf{D}_{\ell} = \begin{bmatrix} d_{\ell} & -z^{-1} \\ -z^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & z^{-1} \\ z^{-1} & d_{\ell} \end{bmatrix} \tag{29}$$

We can state the obtained result as a theorem.

Theorem 1: Given a set of polyphase filters of length m with non-zero coefficients and

$$\mathbf{K}_{\ell}^{(i)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{G}_{\ell}^{(i)} = \frac{z^{-1}(-z^{2})^{-s}}{2M} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; M \leq d < 2M, \; 0 \leq \ell < d-M, \; \ell \neq \frac{d-M}{2} \tag{30}$$

then the following realization for $\mathbf{G}_{\ell}$ and $\mathbf{K}_{\ell}$ always exists:

$$\mathbf{G}_{\ell}^{(i)} = \prod_{i=i_2+1}^{i_3} \mathbf{C}_{\ell,2i}\mathbf{C}_{\ell,2i+1} \cdot \prod_{i=i_1+1}^{i_2} (\mathbf{C}_{\ell,i}\mathbf{B}_{\ell,i}) \cdot \prod_{i=i_0+1}^{i_1} (\mathbf{A}_{\ell,i}\mathbf{C}_{\ell,i}) \cdot \prod_{i=1}^{i_0} (\mathbf{A}_{\ell,i}\mathbf{B}_{\ell,i}) \cdot \mathbf{G}_{\ell,0} \tag{31}$$

$$\mathbf{K}_{\ell}^{(i)} = \mathbf{K}_{\ell,0}^{(i)} \cdot \prod_{i=i_0}^{1} (\mathbf{B}_{\ell,i}^{-1}\mathbf{A}_{\ell,i}^{-1}) \prod_{i=i_1}^{i_0+1} (-z^{-2}\mathbf{C}_{\ell,i}^{-1}\mathbf{A}_{\ell,i}^{-1}) \prod_{i=i_2}^{i_1+1} (-z^{-2}\mathbf{B}_{\ell,i}^{-1}\mathbf{C}_{\ell,i}^{-1}) \prod_{i=i_3}^{i_2+1} (-z^{-4}\mathbf{C}_{\ell,2i+1}^{-1}\mathbf{C}_{\ell,2i}^{-1}) \tag{32}$$

with $s = 2i_3 - i_0$ and $m = i_3 + 1$. The total number of Zero-Delay and Maximum-Delay matrices is identical for all ℓ. However, the order of the matrices can be chosen individually for each ℓ. The completeness of the factorization is shown in Appendix A.

Case (ii), $\quad d - M < \ell < M$. The derivation for ℓ in the range $d-M < \ell < M$ is very similar to the one described above and will therefore only be shortly sketched. Again, we just regard the range $d-M < \ell \leq \frac{d}{2}$, because the other half of the range can be obtained easily when substituting ℓ by $d - M - \ell$.
The modulation matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ here satisfy

$$[\mathbf{C}_1]_{k,M+d-\ell} = (-1)^{s-1}[\mathbf{C}_1]_{k,\ell} \qquad [\mathbf{C}_2]_{k,M-1-\ell} = (-1)^{s}[\mathbf{C}_2]_{k,2M-1-d+\ell} \tag{33}$$

$$[\mathbf{C}_1]_{k,\ell+M} = (-1)^{s-1}[\mathbf{C}_1]_{k,d-\ell} \qquad [\mathbf{C}_2]_{k,M-1-d+\ell} = (-1)^{s}[\mathbf{C}_2]_{k,2M-1-\ell} \tag{34}$$

We see that always two columns of these matrices are identical up to the sign, such that the submatrices of the analysis and synthesis polyphase matrices can be written as:

$$\mathbf{E}_\ell^{(ii)}(z) = \begin{bmatrix} c^1_{0,\ell} & c^1_{0,d-\ell} \\ c^1_{1,\ell} & c^1_{1,d-\ell} \\ \vdots & \vdots \\ c^1_{M-1,\ell} & c^1_{M-1,d-\ell} \end{bmatrix} \underbrace{\begin{bmatrix} G_\ell(-z^2) & (-1)^{s-1} z^{-1} G_{M+d-\ell}(-z^2) \\ (-1)^{s-1} z^{-1} G_{\ell+M}(-z^2) & G_{d-\ell}(-z^2) \end{bmatrix}}_{:=\mathbf{G}_\ell^{(ii)}} \tag{35}$$

and

$$\mathbf{R}_\ell^{(ii)}(z) = \underbrace{\begin{bmatrix} K_{d-\ell}(-z^2) & (-1)^s z^{-1} K_{M+d-\ell}(-z^2) \\ (-1)^s z^{-1} K_{\ell+M}(-z^2) & K_\ell(-z^2) \end{bmatrix}}_{:=\mathbf{K}_\ell^{(ii)}} \begin{bmatrix} c^2_{0,2M-1-d+\ell} & \dots & c^2_{M-1,2M-1-d+\ell} \\ c^2_{0,2M-1-\ell} & \dots & c^2_{M-1,2M-1-\ell} \end{bmatrix} \tag{36}$$

The matrices $\mathbf{G}_\ell^{(ii)}$ and $\mathbf{K}_\ell^{(ii)}$ are slightly different from the ones in case (i). In $\mathbf{G}_\ell^{(i)}$ the delays z^{-1} were placed in the lower row of the matrix and in $\mathbf{G}_\ell^{(ii)}$ they are placed on the anti-diagonal. Furthermore, the sign on the anti-diagonal is now the same. Comparing the result for the product $\mathbf{K}_\ell^{(ii)}\mathbf{G}_\ell^{(ii)}$ with the PR constraints (14) and (16), we obtain that the following relationship has to be satisfied for perfect reconstruction:

$$\mathbf{K}_\ell^{(ii)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{G}_\ell^{(ii)} = \frac{(-z^{-2})^s}{2M} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad d - M < \ell < 2M \tag{37}$$

Due to the different form of the matrices $\mathbf{G}_\ell^{(ii)}$ and $\mathbf{K}_\ell^{(ii)}$ (compared to case (i)), we also have to use slightly different forms for the starting-point matrices $\mathbf{G}_{\ell,0}^{(ii)}$ and $\mathbf{K}_{\ell,0}^{(ii)}$ containing length one polyphase components. Assuming $\mathbf{G}_{\ell,0}^{(ii)}$ and its inverse to be

$$\mathbf{G}_{\ell,0}^{(ii)} = \begin{bmatrix} g_0 & z^{-1} g_1 \\ z^{-1} g_2 & g_3 \end{bmatrix}, \quad (\mathbf{G}_{\ell,0}^{(ii)})^{-1} = \frac{1}{g_0 g_3 - z^{-2} g_1 g_2} \begin{bmatrix} g_3 & -g_1 z^{-1} \\ -g_2 z^{-1} & g_0 \end{bmatrix} \tag{38}$$

$\mathbf{K}_{\ell,0}^{(ii)}$ has to be derived from $(\mathbf{G}_{\ell,0}^{(ii)})^{-1}$ by adding a possible delay and a scaling by $1/2M$. Restricting ourselves to the case where all filters in $\mathbf{K}_{\ell,0}^{(ii)}$ are causal and FIR, we see from (38) that at least one of the coefficients in $\mathbf{G}_{\ell,0}^{(ii)}$ has to be zero. Thus, we obtain the following solutions for $\mathbf{K}_{\ell,0}^{(ii)}$:

$$\mathbf{K}_{\ell,0}^{(ii)} = \frac{1}{2M g_1 g_2} \begin{bmatrix} g_3 & -g_1 z^{-1} \\ -g_2 z^{-1} & g_0 \end{bmatrix}, \ \mathbf{K}_{\ell,0}^{(ii)} \mathbf{G}_{\ell,0}^{(ii)} = \begin{bmatrix} -z^{-2} & 0 \\ 0 & -z^{-2} \end{bmatrix} \quad \text{if } g_0 = 0 \text{ or } g_3 = 0 \tag{39}$$

$$\mathbf{K}_{\ell,0}^{(ii)} = \frac{1}{2M g_0 g_3} \begin{bmatrix} g_3 & -g_1 z^{-1} \\ -g_2 z^{-1} & g_0 \end{bmatrix}, \ \mathbf{K}_{\ell,0}^{(ii)} \mathbf{G}_{\ell,0}^{(ii)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{if } g_1 = 0 \text{ or } g_2 = 0 \tag{40}$$

In order to increase the filter length and the delay, the same Zero-Delay matrices and Maximum-Delay matrices can be used as described in (26) and (29), respectively.
The case $\ell = d/2$ with d being even needs special consideration. In this case, the matrices $\mathbf{G}_{d/2}^{(ii)}$ and $\mathbf{K}_{d/2}^{(ii)}$ write

$$\mathbf{G}_{d/2}^{(ii)} = \begin{bmatrix} G_{d/2}(-z^2) & (-1)^{s-1}z^{-1}G_{d/2+M}(-z^2) \\ (-1)^{s-1}z^{-1}G_{d/2+M}(-z^2) & G_{d/2}(-z^2) \end{bmatrix} \tag{41}$$

$$\mathbf{K}_{d/2}^{(ii)} = \begin{bmatrix} K_{d/2}(-z^2) & (-1)^{s}z^{-1}K_{d/2+M}(-z^2) \\ (-1)^{s}z^{-1}K_{d/2+M}(-z^2) & K_{d/2}(-z^2) \end{bmatrix} \tag{42}$$

Both matrices are Toeplitz and the only possible cascade that keeps this structure using the matrices $\mathbf{G}_{d/2,0}^{(ii)}$ and $\mathbf{K}_{d/2,0}^{(ii)}$ in (38)-(40), as well as the Zero-Delay and Maximum-Delay matrices in (26) and (29), is given by

$$\mathbf{G}_{\ell,0}^{(ii)} = \begin{bmatrix} 0 & g_1 z^{-1} \\ g_1 z^{-1} & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{G}_{\ell,0}^{(ii)} = \begin{bmatrix} g_0 & 0 \\ 0 & g_0 \end{bmatrix} \tag{43}$$

for the starting point and

$$\mathbf{C}_{\ell,i} = \begin{bmatrix} 0 & z^{-1} \\ z^{-1} & 0 \end{bmatrix} \tag{44}$$

for a further increase of the delay.

Case (iii), $\ell = (d-M)/2$. For $\ell = (d-M)/2$, the PR constraints on the polyphase components, as given in (18), do not have the same form as in the other cases discussed above. Always two polyphase components can be chosen arbitrarily. This is due to the fact that the output of this analysis polyphase component will be multiplied with zero in the transform and the synthesis polyphase component is fed with a subband signal which is identical to zero. Thus, these two polyphase ¡filters do not have any influence in the filter bank and can be omitted in the realization, resulting in the lowest implementation cost.
The remaining two filters can only have one non-zero coefficient each in order to satisfy the the PR constraint. Thus, their implementation is very simple.

Note: For reasons of conciseness we restrict ourselves in the following to filter banks with an overall system delay of $D = 2sM + 2M - 1$ samples. In this case, the PR constraints for all polyphase components can be expressed by (14) and (15). Using the modifications discussed in this section, the analysis and synthesis polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ can be written as:

$$\mathbf{E}(z) = \tilde{\mathbf{C}}_1 \mathbf{G}, \qquad \mathbf{R}(z) = \mathbf{K}\tilde{\mathbf{C}}_2^t \tag{45}$$

with

$$[\tilde{\mathbf{C}}_1]_{k,\ell} = [\mathbf{C}_1]_{k,\ell}, \quad [\tilde{\mathbf{C}}_1]_{k,M-1-\ell} = [\mathbf{C}_1]_{k,2M-1-\ell}, \quad 0 \le k < M, \quad 0 \le \ell < M/2 \tag{46}$$

$$[\tilde{\mathbf{C}}_2]_{k,\ell} = [\mathbf{C}_2]_{k,\ell}, \quad [\tilde{\mathbf{C}}_2]_{k,M-1-\ell} = [\mathbf{C}_2]_{k,2M-1-\ell}, \quad 0 \le k < M, \quad 0 \le \ell < M/2 \tag{47}$$

$$\begin{aligned}\mathbf{G} &= \mathrm{diag}([G_0(-z^2),\dots,G_{M/2-1}(-z^2), z^{-1}G_{3M/2}(-z^2),\dots,z^{-1}G_{2M-1}(-z^2)]) \\ &+(-1)^s\mathbf{J}\cdot\mathrm{diag}([-z^{-1}G_M(-z^2),\dots,-z^{-1}G_{3M/2-1}(-z^2), G_{M/2}(-z^2),\dots,G_{M-1}(-z^2)])\end{aligned} \tag{48}$$

$$\begin{aligned}\mathbf{K} &= \mathrm{diag}([z^{-1}K_{2M-1}(-z^2),\dots,z^{-1}K_{3M/2}(-z^2), K_{M/2-1}(-z^2),\dots,K_0(-z^2)]) \\ &+(-1)^s\mathbf{J}\cdot\mathrm{diag}([K_M(-z^2),\dots,K_{3M/2-1}(-z^2), -K_{M/2}(-z^2),\dots,-K_{M-1}(-z^2)])\end{aligned} \tag{49}$$

Note that in this case $\tilde{\mathbf{C}}_1 = (-1)^s\tilde{\mathbf{C}}_2$ and $\tilde{\mathbf{C}}_1^{-1}\sqrt{2M} = \tilde{\mathbf{C}}_1^t$. However, similar results can be obtained for arbitrary delays when taking into consideration the results obtained from (16) and (18).

4 Design of Identical Analysis and Synthesis Filters

Originally, we started this work assuming that we have two different prototype filters: one for the analysis filters and one for the synthesis. However, from the PR constraint (14) we have seen that both prototypes are highly related to each other in the case of a PR filter bank. In fact, the prototypes' polyphase components have to be equal apart from scale factors and possible zero-paddings. For obtaining different analysis and synthesis prototype filters which both have the desired frequency responses, the design freedom is relatively small. Therefore, the designer often only considers the case where analysis and synthesis prototype are identical:

$$K_\ell(z) = G_\ell(z), \quad \ell = 0,\dots,2M-1 \tag{50}$$

In this case, the PR constraint (15) can be expressed by the analysis polyphase filters only:

$$z^{-1}G_\ell(-z^2)G_{d-\ell}(-z^2) + z^{-1}G_{d-\ell-M}(-z^2)G_{\ell+M}(-z^2) = \det(\mathbf{G}_\ell^{(i)}) = \frac{(-z^{-2})^s z^{-1}}{2M} \tag{51}$$

Realizing $\mathbf{G}_\ell^{(i)}$ as in (31), we also know that $\det(\mathbf{G}_\ell^{(i)})$ writes:

$$\det(\mathbf{G}_\ell^{(i)}) = \prod_{i=i_2+1}^{i_3}\det(\mathbf{C}_{\ell,2i}\mathbf{C}_{\ell,2i+1})\prod_{i=i_1+1}^{i_2}\det(\mathbf{C}_{\ell,i}\mathbf{B}_{\ell,i})\prod_{i=i_0+1}^{i_1}\det(\mathbf{A}_{\ell,i}\mathbf{C}_{\ell,i})\prod_{i=1}^{i_0}\det(\mathbf{A}_{\ell,i}\mathbf{B}_{\ell,i})\cdot\det(\mathbf{G}_{\ell,0}^{(i)}) \tag{52}$$

Using the properties $\det(\mathbf{A}_{\ell,i}) = \det(\mathbf{B}_{\ell,i}) = 1$ and $\det(\mathbf{C}_{\ell,i}) = -z^{-2}$, we obtain

$$\det(\mathbf{G}_\ell^{(i)}) = (-z^{-2})^s\det(\mathbf{G}_{\ell,0}^{(i)}) \tag{53}$$

and thus:

$$\det(\mathbf{G}_{\ell,0}^{(i)}) = z^{-1}(g_{0,\ell}g_{3,\ell} - g_{1,\ell}g_{2,\ell}) \stackrel{!}{=} \frac{z^{-1}}{2M} \tag{54}$$

With the following realization for $\mathbf{G}_{\ell,0}^{(i)}$:

$$\mathbf{G}_{\ell,0}^{(i)} = \frac{1}{2M} \begin{bmatrix} 1 & 0 \\ g_{0,\ell}z^{-1} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & g_{1,\ell} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g_{2,\ell} & 1 \end{bmatrix} \tag{55}$$

the relationship (54) is satisfied and we have the advantage that $\mathbf{K}_{\ell,0}^{(i)}$ contains the same coefficients as $\mathbf{G}_{\ell,0}^{(i)}$. It writes

$$\mathbf{K}_{\ell,0}^{(i)} = \frac{1}{2M} \begin{bmatrix} 1 & 0 \\ -g_{2,\ell} & 1 \end{bmatrix} \begin{bmatrix} 1 & -g_{1,\ell} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ -g_{0,\ell}z^{-1} & 1 \end{bmatrix} \tag{56}$$

5 Biorthogonal Cosine-Modulated Filter Banks without DC-Leakage

When processing signals with a DC component (e.g. images), it is important to use filter banks without DC leakage, meaning that the DC component of the input signal only affects the lowpass subband signal. Otherwise, artifacts such as the checkerboard effect may occur when quantizing the subband signals. Figure 5 demonstrates this phenomenon for a gray scale image containing only a DC component. The input signal is split into subbands using two different sets of analysis filters (with and without DC leakage). Then, all subbands apart from the lowpass band, where we expect the signal to be located, are suppressed and the image is reconstructed, resulting once in an image with visible checkerboard artifacts and once in a perfectly reconstructed single color image.

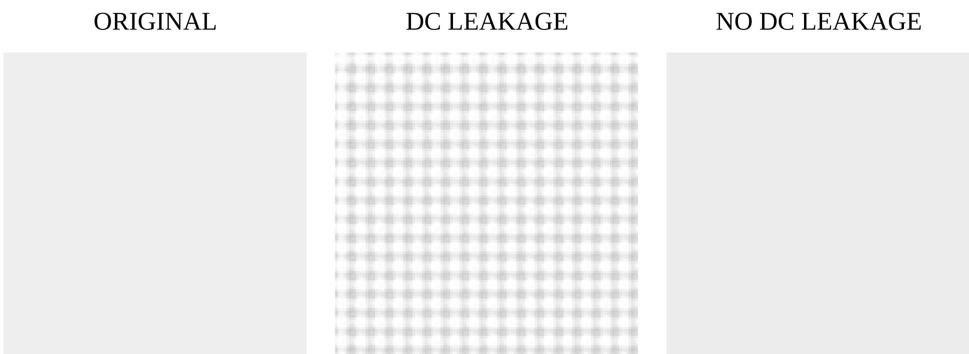

Figure 5: Original image and reconstructed images using filter banks with and without DC leakage

A filter bank is free of DC leakage if all analysis filters apart from the lowpass filter have at least one zero at frequency $\omega = 0$. For the biorthogonal cosine-modulated filter bank this means that $H_k(\omega = 0) = 0$ for $k = 1, \dots, M-1$, while the lowpass filter has to satisfy $H_0(\omega = 0) = 1$.
Let us consider the vector $\mathbf{h}(z) = [H_0(z), \dots, H_{M-1}(z)]^t$, which can be obtained from the analysis polyphase matrix as:

$$\mathbf{h}(z) = \mathbf{E}(z^M)[1, z^{-1}, \dots, z^{-(M-1)}]^t \tag{57}$$

We are interested in the DC behavior ($\omega = 0$ and thus $z = 1$), for which the upper equation writes:

$$[1, 0, \dots, 0]^t = \mathbf{E}(z^M)[1, 1, \dots, 1]^t|_{z=1} \tag{58}$$

Only considering the case $D = 2sM + 2M - 1$, the analysis polyphase matrix has the form in (45). Thus, we get the following linear system of equations:

$$\tilde{\mathbf{C}}_1^{-1}[1, 0, \dots, 0]^t = \mathbf{G}(z)[1, 1, \dots, 1]^t|_{z=1} \tag{59}$$

Splitting the matrix $\mathbf{G}(z)$ into its submatrices $\mathbf{G}_\ell^{(i)}(z)$, $\ell = 0, \dots, M/2-1$, and taking into consideration that $\mathbf{G}_\ell^{(i)}(z)$ can be realized by the cascade (31), equation (59) yields

$$\begin{aligned}\begin{bmatrix}[\tilde{\mathbf{C}}_1^{-1}]_{\ell,0} \\ [\tilde{\mathbf{C}}_1^{-1}]_{d-\ell,0}\end{bmatrix} &= \prod_{i=i_2+1}^{i_3} \begin{bmatrix} c_{\ell,2i} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_{\ell,2i+1} & 1 \\ 1 & 0 \end{bmatrix} \prod_{i=i_1+1}^{i_2} \begin{bmatrix} c_{\ell,i} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b_{\ell,i} \\ 0 & 1 \end{bmatrix} \\ &\prod_{i=i_0+1}^{i_1} \begin{bmatrix} 1 & 0 \\ a_{\ell,i} & 1 \end{bmatrix} \begin{bmatrix} c_{\ell,i} & 1 \\ 1 & 0 \end{bmatrix} \prod_{i=1}^{i_0} \begin{bmatrix} 1 & 0 \\ a_{\ell,i} & 1 \end{bmatrix} \begin{bmatrix} 1 & b_{\ell,i} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} g_{\ell,0} & g_{\ell,1} \\ g_{\ell,2} & g_{\ell,3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\end{aligned} \tag{60}$$

and

$$\begin{aligned}\begin{bmatrix} g_{\ell,0} + g_{\ell,1} \\ g_{\ell,2} + g_{\ell,3} \end{bmatrix} &= \prod_{i=i_0}^{1} \begin{bmatrix} 1 & -b_{\ell,i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a_{\ell,i} & 1 \end{bmatrix} \prod_{i=i_1}^{i_0+1} \begin{bmatrix} 0 & 1 \\ 1 & -c_{\ell,i} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a_{\ell,i} & 1 \end{bmatrix} \\ &\cdot \prod_{i=i_2}^{i_1+1} \begin{bmatrix} 1 & -b_{\ell,i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -c_{\ell,i} \end{bmatrix} \prod_{i=i_3}^{i_2+1} \begin{bmatrix} 0 & 1 \\ 1 & -c_{\ell,2i+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -c_{\ell,2i} \end{bmatrix} \begin{bmatrix} [\tilde{\mathbf{C}}_1^{-1}]_{\ell,0} \\ [\tilde{\mathbf{C}}_1^{-1}]_{2M-1-\ell,0} \end{bmatrix}\end{aligned} \tag{61}$$

Thus, designing filter banks without DC leakage is possible when just imposing the upper constraint on the matrix $\mathbf{G}_{\ell,0}$ which does not reduce the parameter space for the filter optimization a lot.

6 Implementation Cost

In this section, the implementation cost of the direct implementation of the polyphase components is compared to the cost for an implementation by cascading Zero-Delay and Maximum-Delay matrices. For this, note that the direct polyphase filter implementation as shown in Figure 3 needs $2mM$ multiplications and $2(m-1)M$ additions if a prototype filter of length $N = 2mM$ and an overall system delay of $D = 2sM + 2M - 1$ are considered. When realizing the polyphase filters with Zero-Delay and Maximum-Delay matrices, we need $M/2$ matrices $\mathbf{G}_\ell$ in parallel for $\ell = 0, \dots, M/2-1$. The implementation cost for each of these realizations is as follows [KM97a]:

- The first step of the iteration, i.e. the product of the matrix $\mathbf{G}_{\ell,0}$ from (25) with the input samples needs 4 multiplications and 2 additions.
- In order to obtain polyphase filters of length m, the cascade contains $2(m-1)$ Low-Delay or Zero-Delay matrices that each can be realized with 1 multiplication and 1 addition.

Thus, the implementation cost for the filtering part using Zero-Delay and Maximum-Delay matrices is

$$(4 + 2m - 2)M/2 = (m + 1)M \quad \text{multiplications} \quad (62)$$
$$(2 + 2m - 2)M/2 = mM \quad \text{addition} \quad (63)$$

which is approximately half the implementation cost of the original polyphase filtering. Since the same matrix coefficients are found in the analysis and in the synthesis cascade, a coefficient quantization does not change the inherent PR property of the realization. Thus, we have the freedom to optimize the coefficients not only with regard to the frequency response, but also with regard to an efficient hardware (VLSI) implementation. Overall, we see that the implementation via Zero-Delay and Maximum-Delay matrices is well suited for real-time applications, where a low arithmetic cost is required.

7 VLSI Efficient Prototype Realization

For a VLSI realization it is favorable if the multiplications with the matrix or filter coefficients can be replaced by a small number of shift and add operations. In order to obtain such filters, we start with a given PR prototype filter whose polyphase filters can be realized according to (31). Examples for filters with the same overall system delay, but with different lengths, are shown in Figure 6. Note that the stopband attenuation increases with the filter length. This is one of the main advantages of biorthogonal filter banks when compared to paraunitary ones.

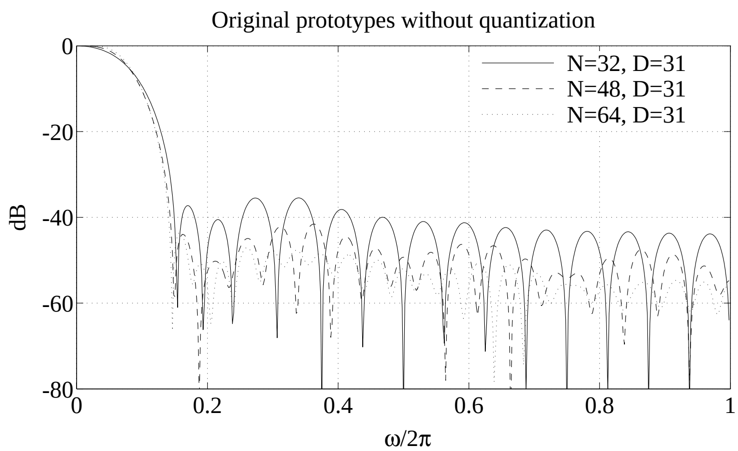

Figure 6: Prototype filters of different lengths for an 8-channel filter bank

The algorithm to obtain VLSI-efficient prototypes is as follows: In a first step, the coefficients in (31) are quantized according to a given coefficient wordlength using a signed binary number representation. Then, we iteratively replace the least sensitive coefficient by a coarser approximation, which reduces the number of additions being necessary for that coefficient by one (the least sensitive coefficient is defined as the one that results in the smallest increase of the cost function when being replaced by the coarser approximation). This pruning is continued as long as the complexity in terms of shift and add operations is above the desired one, or as long as the frequency response does not change significantly, depending on which constraint (complexity or frequency response) is more important.

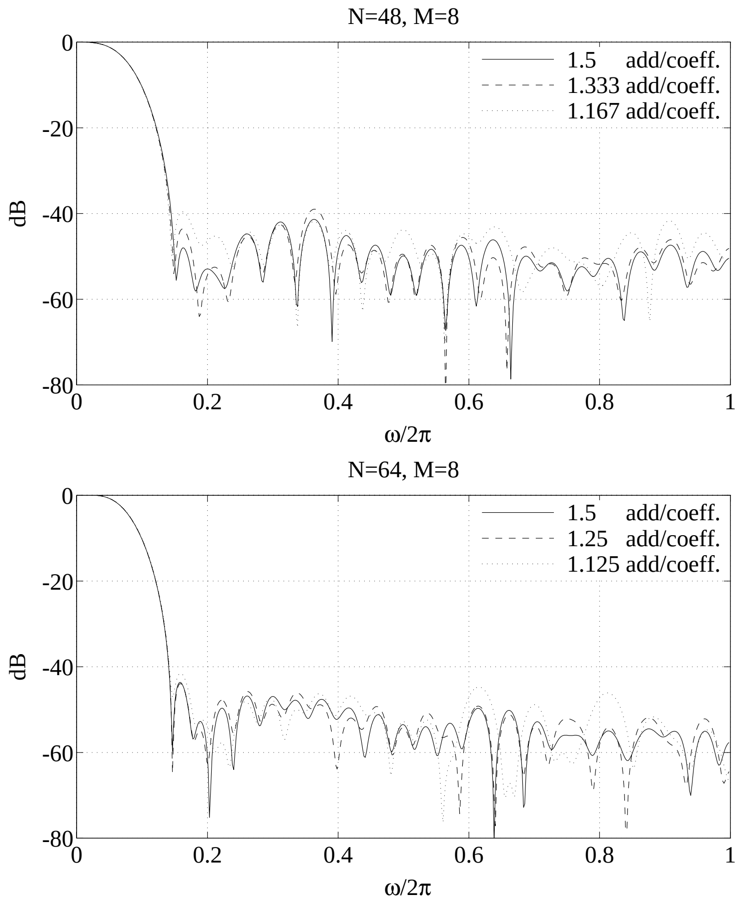

Figure 7: Magnitude responses of prototype realizations with different complexities

Figure 7 shows the magnitude responses of prototypes with different complexities. We see that reducing the complexity results in a filter with a worse stopband attenuation. However, when allowing 1.167 additions per coefficient in average for the prototype filter of length $N = 48$, we still obtain a stopband attenuation of 40 dB. In several examples it has turned out that longer prototypes are more sensitive to coefficient quantization than shorter ones. Thus, longer prototypes typically require a larger average complexity per coefficient. On the other hand, one also obtains a higher stopband attenuation, as can be seen in Figure 7 when comparing the magnitude responses of the length-48 and length-64 prototypes.

Figure 8 shows the magnitude responses of a length-48 and a length-64 prototype filter that both require a total amount of 80 additions and yield an overall system delay of 31 samples. It turns out that the length-64 prototype filter has a higher stopband attenuation although the number of additions per filter coefficient is lower than for the length-48 prototype (1.25 add/coeff. versus 1.67 add/coeff.). This is due to the fact that the frequency response in the case $N = 48$ cannot be better than the one given in Figure 6 for the unquantized case.

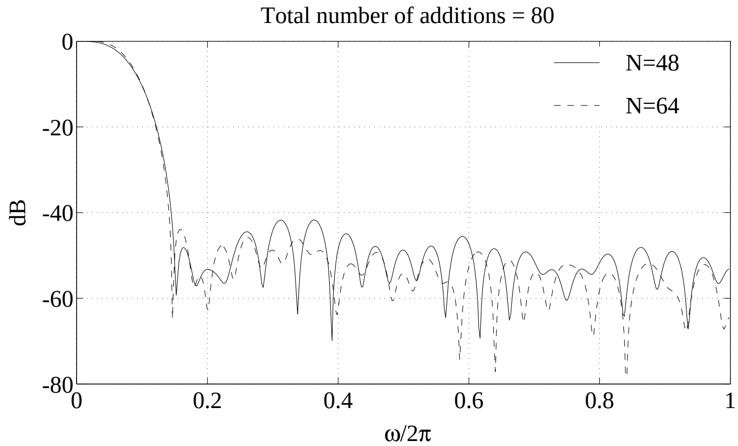

Figure 8: Comparison of prototype filter realizations with different lengths N but the same complexities

8 Time-Varying Biorthogonal Cosine-Modulated Filter Banks

Most of the real-world signals being treated with filter banks cannot be considered as stationary. In order to improve the coding efficiency of the filter bank it is therefore useful to adapt the filter characteristic and the number of bands to the signal statistics. In [Edl89] it was shown how to switch from one cosine modulated filter bank to another. Later approaches were mostly for non-modulated filter banks [NBS91, NIS94, HKV93, HV94, PV96, dQR93, SNBS95]. We here just describe the basic idea for the design of time-varying modulated filter banks. A detailed description can be found in [Sch97, SK97].

The main idea for the following description of time-varying filter banks is the observation that if a signal first passes a time-varying system or matrix $\mathbf{F}(z, m)$ and then a delay z^{-1}, the output is the same as if the signal is first delayed and then passes the system or matrix at the state of the previous time step:

$$z^{-1} \cdot \mathbf{F}(z, m) = \mathbf{F}(z, m-1) \cdot z^{-1} \tag{64}$$

and thus:

$$z^{-d}\mathbf{I} = z^{-d}\mathbf{F}(z, m)^{-1}\mathbf{F}(z, m) = \mathbf{F}(z, m-d)^{-1} z^{-d}\mathbf{F}(z, m) \tag{65}$$

This fact can be used for the design of time-varying polyphase matrices. If, for example, the matrix $\mathbf{G}_\ell^{(i)}$ writes

$$\mathbf{G}_\ell^{(i)}(z,m) = \prod_{i=i_0+1}^{i_1} \mathbf{A}_{i,\ell}(m)\mathbf{B}_{i,\ell}(m) \prod_{i=1}^{i_0} \mathbf{C}_{i,\ell}(m)\mathbf{G}_{\ell,0}^{(i)}(z,m) \tag{66}$$

then the synthesis matrix $\mathbf{K}_\ell^{(i)}$ has to fulfill

$$\mathbf{K}_\ell(z,m) = (\mathbf{G}_\ell^{(i)}(z,m-2i_0-1))^{-1}z^{-1}\prod_{i=i_0}^{1} \mathbf{C}_{i,\ell}^{-1}(m-2i)z^{-2}\prod_{i=i_1}^{i_0+1} \mathbf{B}_{i,\ell}^{-1}(m)\mathbf{A}_{i,\ell}^{-1}(m) \tag{67}$$

in order to obtain perfect reconstruction of the filter bank also in the time-varying case. How to switch the number of channels is described in [Sch97, SK97].

9 Conclusions

In this paper we have connected two different approaches for the design of biorthogonal cosine-modulated filter banks with perfect reconstruction. Based on the PR constraints we have shown how the polyphase filters can be realized using Zero-Delay and Maximum-Delay matrices. This structure has the advantage that it automatically guarantees PR of the filter bank even after coefficient quantization and is thus suitable for VLSI designs. Furthermore, the implementation cost is nearly halved, when compared to a direct realization of the polyphase filters. Using the factorization into Zero-Delay and Maximum-Delay matrices, we can design different prototype filters for the analysis and synthesis or restrict the first matrix in the cascade such as to obtain one common prototype. Using a modified set of constraints for the first matrix, we can also obtain biorthogonal cosine-modulated filter banks without DC leakage. The extension of the framework to time-varying filter banks has been shortly sketched.

Appendix A: Completeness of the Factorization

In the following, we show that all PR prototypes whose polyphase components satisfy the PR constraint (24) can be realized using the cascade described in (31) and (32).
Let us assume that the matrices $\mathbf{G}_\ell^{(i)}$ and $\mathbf{K}_\ell^{(i)}$ have been obtained from matrices $\mathbf{G}_{\ell,k}^{(i)}$ and $\mathbf{K}_{\ell,k}^{(i)}$ by introduction of a Maximum-Delay matrix:

$$\mathbf{G}_\ell^{(i)} = \mathbf{C}_\ell \mathbf{G}_{\ell,k}^{(i)}, \qquad \mathbf{K}_\ell^{(i)} = \mathbf{K}_{\ell,k}^{(i)}\mathbf{C}_\ell^{-1}(-z^{-2}) \tag{68}$$

Then, the matrices $\mathbf{G}_{\ell,k}^{(i)}$ and $\mathbf{K}_{\ell,k}^{(i)}$ can be calculated as $\mathbf{G}_{\ell,k}^{(i)} = \mathbf{C}_\ell^{-1}\mathbf{G}_\ell^{(i)}$ and $\mathbf{K}_{\ell,k}^{(i)} = \mathbf{K}_\ell^{(i)}(-z^2)\mathbf{C}_\ell$ and write:

$$\mathbf{G}_{\ell,k}^{(i)} = \begin{bmatrix} (-1)^{s-1}G_{\ell+M}(-z^2) & G_{d-\ell}(-z^2) \\ zG_\ell(-z^2) - (-1)^{s-1}zG_{\ell+M}(-z^2)c_\ell & (-1)^s zG_{d-\ell-M}(-z^2) - c_\ell z G_{d-\ell}(-z^2) \end{bmatrix} \quad (69)$$

$$\mathbf{K}_{\ell,k}^{(i)} = \begin{bmatrix} -c_\ell z K_{d-\ell}(-z^2) - (-1)^{s-1}zK_{d-\ell-M}(-z^2) & -K_{d-\ell}(-z^2) \\ -c_\ell z(-1)^s K_{\ell+M}(-z^2) - zK_\ell(-z^2) & (-1)^s K_{\ell+M}(-z^2) \end{bmatrix} \quad (70)$$

For the matrices to be causal, we obtain the following constraints on c_ℓ:

$$g_\ell(0) - c_\ell(-1)^{s-1}g_{\ell+M}(0) = 0 \qquad (-1)^s k_{d-\ell-M}(0) - c_\ell k_{d-\ell}(0) = 0 \quad (71)$$

$$(-1)^s g_{d-\ell-M}(0) - c_\ell g_{d-\ell}(0) = 0 \qquad k_\ell(0) - c_\ell(-1)^{s-1}k_{\ell+M}(0) = 0 \quad (72)$$

When combining them, the following time domain formulations of the PR constraints in (14) and (15) arise for $s \neq 0$:

$$g_\ell(0)k_{d-\ell}(0) + g_{\ell+M}(0)k_{d-\ell-M}(0) = 0 \qquad k_\ell(0)g_{d-\ell}(0) + k_{\ell+M}(0)g_{d-\ell-M}(0) = 0 \quad (73)$$

$$g_\ell(0)k_{\ell+M}(0) - g_{\ell+M}(0)k_\ell(0) = 0 \qquad g_{d-\ell}(0)k_{d-\ell-M}(0) - g_{d-\ell-M}(0)k_{d-\ell}(0) = 0 \quad (74)$$

The PR constraint for $\mathbf{G}_{\ell,k}^{(i)}$ and $\mathbf{K}_{\ell,k}^{(i)}$ now writes

$$\mathbf{K}_{\ell,k}^{(i)}\mathbf{G}_{\ell,k}^{(i)} = \frac{(-z^{-2})^{s-1}z^{-1}}{2M} \quad (75)$$

where the delay parameter s has been reduced by one. This procedure can be continued as long as $s - k > 0$. Then, in a second step, we use Zero-Delay matrices in order to further decrease the filter length. Similar to the application of Maximum-Delay matrices described above, one can show that the constraints that arise on the values a_ℓ and b_ℓ are again nothing else but a time domain formulation of the PR constraints, see also [DS96, KM97b].

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